

A NON-HOMOGENEOUS ORBIT CLOSURE OF A DIAGONAL SUBGROUP

FRANÇOIS MAUCOURANT

ABSTRACT. Let $G = \mathbf{SL}(n, \mathbf{R})$ with $n \geq 6$. We construct examples of lattices $\Gamma \subset G$, subgroups A of the diagonal group D and points $x \in G/\Gamma$ such that the closure of the orbit Ax is not homogeneous and such that the action of A does not factor through the action of a one-parameter non-unipotent group. This contradicts a conjecture of Margulis.

1. INTRODUCTION

1.1. Topological rigidity and related questions. Let G be a real Lie group, Γ a lattice in G , meaning a discrete subgroup of finite covolume, and A a closed connected subgroup. We are interested in the action of A on G/Γ by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [6] and [3] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset $X \subset G$ is said to be *homogeneous* if there exists a closed connected subgroup $H \subset G$ such that $X = Hx$ for some (and hence every) $x \in X$. Let us say that the action of A on G/Γ is *topologically rigid* if for any $x \in G/\Gamma$, the closure \overline{Ax} of the orbit Ax is homogeneous.

The most basic example of a topologically rigid action is when $G = \mathbf{R}^n$, $\Gamma = \mathbf{Z}^n$, A any vector subspace of G . It turns out that the behavior of elements of A for the adjoint action on the Lie algebra \mathfrak{g} of G plays an important role for our problem. Recall that an element $g \in G$ is said to be **Ad**-unipotent if **Ad**(g) is unipotent, and **Ad**-split over \mathbf{R} if **Ad**(g) is diagonalizable over \mathbf{R} . If the closed, connected subgroup A of G is generated by **Ad**-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of A is always topologically rigid, settling a conjecture due to Raghunathan.

2000 *Mathematics Subject Classification.* 22E40 37B99.

When A is generated by elements which are **Ad**-split over \mathbf{R} , much less is known. Consider the model case of $G = \mathbf{SL}(n, \mathbf{R})$ and A the group of diagonal matrices with nonnegative entries. If $n = 2$, it is easy to produce non-homogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when A is a one-parameter subgroup of the diagonal group (see [6], 4.1). However, for A the full diagonal group, if $n \geq 3$, to the best of our knowledge, the only nontrivial example of a nonhomogeneous A -orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice Γ of $G = \mathbf{SL}(3, \mathbf{R})$ and a point $x \in G/\Gamma$ such that for the full diagonal group A , the orbit closure \overline{Ax} is not homogeneous. Her construction was based on the following property of the lattice: there exists a $\gamma \in \Gamma \cap A$ such that the centralizer $C_G(\gamma)$ of γ is isomorphic to $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{R}^*$, and such that $C_G(\gamma) \cap \Gamma$ is, in this product decomposition and up to finite index, $\Gamma_0 \times \langle \gamma \rangle$, where Γ_0 is a lattice in $\mathbf{SL}(2, \mathbf{R})$ (see [4], [7]). Thus in this case the action of A on $C_G(\gamma)/C_G(\gamma) \cap \Gamma$ factors to the action of a 1-parameter non-unipotent subgroup on $\mathbf{SL}(2, \mathbf{R})/\Gamma_0$, which, as we saw, has many non-homogeneous orbits.

Rees' example shows that factor actions of 1-parameter non-**Ad**-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, conjecture 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

Conjecture 1. *Let G be a connected Lie group, Γ a lattice in G , and A a closed, connected subgroup of G generated by **Ad**-split over \mathbf{R} elements. Then for any $x \in G/\Gamma$, one of the following holds :*

- (a) \overline{Ax} is homogeneous, or
- (b) *There exists a closed connected subgroup F of G and a continuous epimorphism ϕ of F onto a Lie group L such that*
 - $A \subset F$,
 - Fx is closed in G/Γ ,
 - $\phi(F_x)$ is closed in L , where F_x denotes the stabilizer $\{g \in F | gx = x\}$,
 - $\phi(A)$ is a one-parameter subgroup of L containing no nontrivial **Ad** _{L} -unipotent elements.

A first step toward this conjecture has been done by Lindenstrauss and Weiss [7], who proved that in the case $G = \mathbf{SL}(n, \mathbf{R})$ and A the full diagonal group, if the closure of a A -orbit contains a compact A -orbit that satisfy some irrationality conditions, then this closure is homogeneous. See also [15]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved that if moreover $\Gamma = \mathbf{SL}(n, \mathbf{Z})$, then the set of bounded A -orbits has Hausdorff dimension $n - 1$ [3, Theorem 10.2].

1.2. Statement of the results. In this article we exhibit some counterexamples to the above conjecture when $G = \mathbf{SL}(n, \mathbf{R})$ for $n \geq 6$ and A is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let D be

the diagonal subgroup of G ; note that D has dimension $n - 1$. Our main result is:

Theorem 1. *Assume $n \geq 6$.*

- (1) *There exists a $(n - 3)$ dimensional closed and connected subgroup A of D , and a point $x \in \mathbf{SL}(n, \mathbf{R})/\mathbf{SL}(n, \mathbf{Z})$ such that the closure of the A -orbit of x satisfies neither condition (a) nor condition (b) of the conjecture.*
- (2) *There exists a lattice Γ of $\mathbf{SL}(n, \mathbf{R})$, a $(n - 2)$ dimensional closed and connected subgroup A of D and a point $x \in \mathbf{SL}(n, \mathbf{R})/\Gamma$ such that the closure of the A -orbit of x satisfies neither condition (a) nor condition (b) of the conjecture.*

It will be clear from the proofs that these examples however satisfy a third condition:

- (c) *There exists a closed connected subgroup F of G and two continuous epimorphisms ϕ_1, ϕ_2 of F onto Lie groups L_1, L_2 such that*
 - $A \subset F$,
 - Fx is closed in G/Γ ,
 - For $i = 1, 2$, $\phi_i(F_x)$ is closed in L_i ,
 - $(\phi_1, \phi_2) : F \rightarrow L_1 \times L_2$ is surjective
 - $(\phi_1, \phi_2) : A \rightarrow \phi_1(A) \times \phi_2(A)$ is not surjective.

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

1.3. Toral endomorphisms. To conclude this introduction, we would like to mention that the idea behind this construction can be also used to yield examples of 'non-homogeneous' orbits for diagonal toral endomorphisms.

Let $1 < p_1 < \dots < p_q$, with $q \geq 2$, be integers generating a multiplicative non-lacunary semigroup of \mathbf{Z} (that is, the \mathbf{Q} -subspace $\oplus_{1 \leq i \leq q} \mathbf{Q} \log(p_i)$ has dimension at least 2). We consider the abelian semigroup Ω of endomorphisms of the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ generated by the maps $z \mapsto p_i z \bmod \mathbf{Z}^n$, $1 \leq i \leq q$.

In the one-dimensional situation, described by Furstenberg [5], every Ω -orbit is finite or dense. If $n \geq 2$, Berend [1] showed that minimal sets are the finite orbits of rational points, but there are others obvious closed Ω -invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integer Hausdorff dimension.

Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if $q = n = 2$, and if $\alpha \in T^1$ is irrational with non-dense p_1 -orbit, the orbit closure of the point $(\alpha, 1/p_2) \in T^2$ is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).

With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for $n \geq 2q$, as the following example shows.

Theorem 2. *Let N be an integer greater than $q \frac{\log p_q}{\log p_1}$, and let z be the point in the $2q$ -dimensional torus T^{2q} defined by the coordinates modulo 1:*

$$z = (z_1, \dots, z_{2q}) = \left(\sum_{k \geq 1} p_1^{-N^{2k}}, \dots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k+1}}, \dots, \sum_{k \geq 1} p_q^{-N^{2k+1}} \right).$$

Then the point $z \in T^{2q}$ is not contained in any rational affine subspace, but its orbit Ωz is not dense.

The proof of Theorem 2 will be the subject of Section 4.

2. SKETCH OF PROOF OF THEOREM 1

2.1. The direct product setup. We now describe how these examples are built. Choose two integers $n_1 \geq 3$, $n_2 \geq 3$, such that $n_1 + n_2 = n$. For $i = 1, 2$, let Γ_i be a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$.

Let g_i be an element of G_i such that $g_i \Gamma_i g_i^{-1}$ intersects the diagonal subgroup D_i of $\mathbf{SL}(n_i, \mathbf{R})$ in a lattice, in other words $g_i \Gamma_i$ has a compact D_i -orbit; such elements exist, see [11]. In fact, we will need an additional assumption on g_i , namely that the tori $g_i^{-1} D_i g_i$ are *irreducible over \mathbf{Q}* . The precise definition of this property and the proof of the existence of such a g_i , a consequence of a theorem of Prasad and Rapinchuk [12, Theorem 1], will be the subject of Section 3.1.

Let $\pi_i : G_i \rightarrow G_i / \Gamma_i$ be the canonical quotient map. Define for $i = 1, 2$:

$$y_i = \pi_i \left(\begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} g_i \right).$$

The D_i -orbit of y_i is dense, by the following argument. It is easily seen that the closure of $D_i y_i$ contains the compact D_i -orbit $\mathcal{T}_i = \pi_i(D_i g_i)$. The \mathbf{Q} -irreducibility of \mathcal{T}_i is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [7, Theorem 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group $H_i < G_i$ such that $H_i y_i = \overline{D_i y_i}$. Again because

of \mathbf{Q} -irreducibility, the group H_i is necessarily the full group, i.e. $H_i = G_i$ (proof of Lemma 3.2)¹.

Let A_1 be the $(n - 3)$ dimensional subgroup of $G_1 \times G_2$ given by:

$$(1) \quad A_1 = \left\{ (diag(a_1, \dots, a_{n_1}), diag(b_1, \dots, b_{n_2})) : \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1 b_1}{a_{n_1} b_{n_2}} = 1, a_i > 0, b_j > 0 \right\}.$$

Then the A_1 -orbit of (y_1, y_2) is not dense in $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ (Lemma 3.3), but $G_1 \times G_2$ is the smallest closed connected subgroup F of $G_1 \times G_2$ such that $\overline{A_1(y_1, y_2)} \subset F(y_1, y_2)$ (Lemma 3.7).

This yields a counterexample to Conjecture 1 which can be summarized as follows:

Proposition 1. *For $i = 1, 2$, let $n_i \geq 3$ and Γ_i be a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$. For A_1, y_1, y_2 depicted as above, the A_1 -orbit of (y_1, y_2) in $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ satisfies neither condition (a) nor condition (b) of Conjecture 1.*

2.2. Theorem 1, part (1). In order to obtain the first part of Theorem 1, choose $\Gamma_i = \mathbf{SL}(n_i, \mathbf{Z})$, $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ and consider the embedding of $G_1 \times G_2$ in G , where matrices are written in blocks:

$$(2) \quad \Psi : (M_{n_1, n_1}, N_{n_2, n_2}) \mapsto \begin{bmatrix} M_{n_1, n_1} & 0_{n_1, n_2} \\ 0_{n_2, n_1} & N_{n_2, n_2} \end{bmatrix}.$$

This embedding gives rise to an embedding $\overline{\Psi}$ of $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ into G / Γ . Let y_1, y_2 be two points as above, let $x = \overline{\Psi}(y_1, y_2)$ and take $A = \Psi(A_1)$. We claim that this point x and this group A satisfy Theorem 1, part (1). In fact, since the image of $\overline{\Psi}$ is a closed connected A -invariant subset of $\mathbf{SL}(n, \mathbf{R}) / \mathbf{SL}(n, \mathbf{Z})$, everything takes place in this direct product.

2.3. Theorem 1, part (2). The second part of Theorem 1 is obtained as follows. Let σ be the nontrivial field automorphism of the quadratic extension $\mathbf{Q}(\sqrt[4]{2}) / \mathbf{Q}(\sqrt{2})$. Consider for any $m \geq 1$:

$$\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma) = \left\{ M \in \mathbf{SL}(m, \mathbf{Z}[\sqrt[4]{2}]) : ({}^t M^\sigma) M = I_m \right\}.$$

Then $\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ is a lattice in $\mathbf{SL}(m, \mathbf{R})$, as will be proved in Section 3.5 (see [4, Appendix] for $m = 3$). Define for $i = 1, 2$, $\Gamma_i = \mathbf{SU}(n_i, \mathbf{Z}[\sqrt[4]{2}], \sigma)$, and $\Gamma = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$. Now consider the map:

$$\varphi : G_1 \times G_2 \times \mathbf{R} \rightarrow G,$$

¹The reader only interested in the case $n = 6$ and $\Gamma = \mathbf{SL}(6, \mathbf{Z})$ might note that when $\Gamma_1 = \Gamma_2 = \mathbf{SL}(3, \mathbf{Z})$, [7, Corollary 1.4] can be used directly in the proof of Lemma 3.2; then the notion of \mathbf{Q} -irreducibility becomes unnecessary, and the entire Section 3.1 can be skipped.

$$(X, Y, t) \mapsto \begin{bmatrix} e^{n_2 t} X & 0 \\ 0 & e^{-n_1 t} Y \end{bmatrix}.$$

Define M to be the image of φ . This time, φ factors into a finite covering $\overline{\varphi}$ of homogeneous spaces:

$$\overline{\varphi} : G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z} \rightarrow M/M \cap \Gamma \subset G/\Gamma,$$

where $\alpha = (3 + 2\sqrt{2}) + \sqrt[4]{2}(2 + 2\sqrt{2})$ satisfies $\alpha^{-1} = \sigma(\alpha)$. Consider the points y_i constructed above, and let $x = \overline{\varphi}(y_1, y_2, 0)$. Choose:

$$A = \left\{ \text{diag}(a_1, \dots, a_n) \mid \prod_{i=1}^n a_i = \frac{a_1 a_{n_1+1}}{a_{n_1} a_n} = 1, a_i > 0 \right\} \subset \mathbf{SL}(n, \mathbf{R}).$$

We claim that this lattice Γ , this point x and this group A satisfy Theorem 1, part (2). What happens here is that the A -orbit of x is a circle bundle over an A_1 -orbit (up to the finite cover $\overline{\varphi}$), like in Rees' example.

3. PROOF OF THEOREM 1

3.1. \mathbf{Q} -irreducible tori. Fix $i \in \{1, 2\}$. Recall that Γ_i is a lattice in $G_i = \mathbf{SL}(n_i, \mathbf{R})$. Since $n_i \geq 3$, by Margulis's arithmeticity Theorem [16, Theorem 6.1.2], there exists a semisimple algebraic \mathbf{Q} -group \mathbf{H}_i and a surjective homomorphism θ from the connected component of identity of the real points of this group $\mathbf{H}_i^0(\mathbf{R})$ to $\mathbf{SL}(n_i, \mathbf{R})$, with compact kernel, such that $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$ is commensurable with Γ_i .

Following Prasad and Rapinchuk, we say that a \mathbf{Q} -torus $\mathbf{T} \subset \mathbf{H}_i$ is \mathbf{Q} -irreducible if it does not contain any proper subtorus defined over \mathbf{Q} . By [12, Theorem 1, (ii)], there exists a maximal \mathbf{Q} -anisotropic \mathbf{Q} -torus $\mathbf{T}_i \subset \mathbf{H}_i$, which is \mathbf{Q} -irreducible. Because any two maximal \mathbf{R} -tori of $\mathbf{SL}(n_i, \mathbf{R})$ are \mathbf{R} -conjugate, there exists $g_i \in G_i$ such that $\theta(\mathbf{T}_i^0(\mathbf{R})) = g_i^{-1} D_i g_i$. The subgroup $\mathbf{T}_i(\mathbf{Z})$ is a cocompact lattice in $\mathbf{T}_i(\mathbf{R})$ since \mathbf{T}_i is \mathbf{Q} -anisotropic [2, Theorem 8.4 and Definition 10.5]. Because $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$ and Γ_i are commensurable and θ has compact kernel, it follows that both $\Gamma_i \cap g_i^{-1} D_i g_i$ and $\theta(\mathbf{T}_i^0(\mathbf{Z})) \cap \Gamma_i \cap g_i^{-1} D_i g_i$ are also cocompact lattices in $g_i^{-1} D_i g_i$. The resulting topological torus $\pi_i(D_i g_i) \subset G_i/\Gamma_i$ will be denoted \mathcal{T}_i . Write $z_i = \pi_i(g_i)$, so that $\mathcal{T}_i = D_i z_i$.

For every $1 \leq k < l \leq n_i$, define as in [7]:

$$N_{k,l}^{(i)} = \left\{ \text{diag}(a_1, \dots, a_{n_i}) : \prod_{s=1}^{n_i} a_s = 1, a_k = a_l, a_s > 0 \right\} \subset D_i,$$

Of interest to us amongst the consequences of \mathbf{Q} -irreducibility is the fact that an element of $\Gamma_i \cap g_i^{-1} D_i g_i$ lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

Lemma 3.1. *For every $1 \leq k < l \leq n_i$, and any closed connected subgroup L of positive dimension of $N_{k,l}^{(i)}$, the L -orbit of z_i is not compact.*

Proof. Assume the contrary, that is Lz_i is compact. This implies that $g_i^{-1}Lg_i \cap \Gamma_i$ is a uniform lattice in $g_i^{-1}Lg_i$, so $g_i^{-1}Lg_i \cap \theta(\mathbf{H}_i(\mathbf{Z}))$ is also a uniform lattice. Since L is nontrivial, there exists an element $\gamma \in \mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R})$ of infinite order, such that $g_i\theta(\gamma)g_i^{-1}$ is in L . Note that since θ has compact kernel, $\mathbf{T}_i(\mathbf{Z})$ is a lattice in $\theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$ and is then a subgroup of finite index in $\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}) \cap \theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$, so there exists $n > 0$ such that γ^n belongs to $\mathbf{T}_i(\mathbf{Z})$. Consider the representation:

$$\begin{aligned} \rho : \mathbf{H}_i^0(\mathbf{R}) &\rightarrow \mathbf{GL}(\mathfrak{sl}(n_i, \mathbf{R})), \\ x &\mapsto \mathbf{Ad}(g_i\theta(x)g_i^{-1}). \end{aligned}$$

Recall that $\chi(\text{diag}(a_1, \dots, a_{n_i})) = a_k/a_l$ is a weight of \mathbf{Ad} with respect to D_i , so χ is a weight of ρ with respect to \mathbf{T}_i . By [12, Proposition 1, (iii)], the \mathbf{Q} -irreducibility of \mathbf{T}_i implies that $\chi(\gamma^n) \neq 1$, but this contradicts the fact that $\theta(\gamma^n) \in g_i^{-1}N_{k,l}^{(i)}g_i$. \square

3.2. Contraction and expansion. For real s , denote by $a_i(s)$ the following $n_i \times n_i$ -matrix:

$$a_i(s) = \text{diag}(e^{s/2}, \underbrace{1, \dots, 1}_{n_i - 2 \text{ times}}, e^{-s/2}),$$

and write simply N_i for $N_{1,n_i}^{(i)}$. Write also:

$$h_i(t) = \begin{bmatrix} 1 & 0 & \dots & 0 & t \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

Then the following commutation relation holds:

$$a_i(s)h_i(t) = h_i(e^st)a_i(s),$$

that is the direction h_i is expanded for positive s ; note that both h_i and a_i commute with elements of N_i . It is easy to check from Equation (1) that

$$A_1 = \{(a_1(s)d_1, a_2(-s)d_2) : s \in \mathbf{R}, d_i \in N_i, i = 1, 2\}.$$

Recall that $y_i = h_i(1)z_i$.

- Lemma 3.2.** (1) *If $s \leq 0$, for any $d \in N_i$ the point $a_i(s)dy_i$ lies in the compact set $K_i = h_i([0, 1])\mathcal{T}_i$.*
 (2) *The D_i -orbit of y_i is dense in G_i/Γ_i .*
 (3) *The set $\{a_i(s)dy_i : s \geq 0, d \in N_i\}$ is dense in G_i/Γ_i .*

Proof. The first statement is clear from the commutation relation. It also implies that $D_i y_i$ contains the compact torus \mathcal{T}_i in its closure.

To prove the second point, we rely heavily on the paper of Lindenstrauss and Weiss. [7, Theorem 1.1] applies here, since the hypotheses of their Theorem is precisely the conclusion of Lemma 3.1 for $L = N_{k,l}^{(i)}$. So the following holds: there exists a reductive subgroup H_i , containing D_i , such that $\overline{D_i y_i} = H_i y_i$, and $H_i \cap \Gamma_i$ is a lattice in H_i . Write $L = D_i \cap C_{G_i}(H_i)$.

Since $D_i y_i$ is not closed, $H_i \neq D_i$, so there exists a nontrivial root relatively to D_i for the Adjoint representation of H_i on its Lie algebra, which is a subalgebra of $\mathfrak{sl}(n_i, \mathbf{R})$. Thus there exist k, l such that $L \subset N_{k,l}^{(i)}$. By [7, step 4.1 of Lemma 4.2], $L z_i$ is compact, so by Lemma 3.1, L is trivial. By [7, Proposition 3.1], H_i is the connected component of the identity of $C_{G_i}(L)$, so $H_i = G_i$, as desired.

The third claim follows from the first and second claim together with the fact that K_i has empty interior. \square

3.3. Topological properties of the A_1 -orbit.

Lemma 3.3. *The A_1 -orbit of (y_1, y_2) is not dense in $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$.*

Proof. Consider the open set $U = K_1^c \times K_2^c$. We claim that the A_1 -orbit of (y_1, y_2) does not intersect U . Indeed, if $(a_1(s)d_1, a_2(-s)d_2) \in A_1$ with $s \in \mathbf{R}$ and $d_i \in N_i$, the previous Lemma implies that if $s \geq 0$, $a_2(-s)d_2 y_2 \in K_2$, and if $s \leq 0$, $a_1(s)d_1 y_1 \in K_1$. \square

The following elementary result will be useful:

Lemma 3.4. *Let $p_i : G_1 \times G_2 \rightarrow G_i$ be the first (resp. second) coordinate morphism. If $F \subset G_1 \times G_2$ is a subgroup such that $p_i(F) = G_i$ for $i = 1, 2$, and $A_1 \subset F$, then $F = G_1 \times G_2$.*

Proof. Let $F_1 = \text{Ker}(p_1) \cap F$. Since F_1 is normal in F , $p_2(F_1)$ is normal in $p_2(F) = G_2$. Note that $N_2 \subset p_2(A_1 \cap \text{Ker}(p_1)) \subset p_2(F_1)$ is not finite, and that G_2 is almost simple, consequently the normal subgroup $p_2(F_1)$ of G_2 is equal to G_2 . Let $(a, b) \in G_1 \times G_2$, by assumption there exists $f \in F$ such that $p_1(f) = a$. Let $f_1 \in F_1$ be such that $p_2(f_1) = b p_2(f)^{-1}$, then $(a, b) = f_1 f \in F$. \square

We will have to apply several times the two following well-known Lemmas:

Lemma 3.5. *Let L be a Lie group, $\Lambda \subset L$ a lattice, M, N two closed, connected subgroups of L , such that for some $w \in L/\Lambda$, Mw and Nw are closed. Then $(M \cap N)w$ is closed.*

Proof. This is a weaker form of [14, Lemma 2.2]. \square

Lemma 3.6. *Let L be a connected Lie group, $\Lambda \subset L$ a discrete subgroup, M, N two subgroups of L , such that M is closed and connected, and N is a countable union of closed sets. For any $w \in L/\Lambda$, if $Mw \subset Nw$, then $M \subset N$.*

Proof. Up to changing Λ by one of its conjugate in L , one can assume that $w = \Lambda \in L/\Lambda$. By assumption, $M\Lambda \subset N\Lambda$ so $M \subset N\Lambda \subset L$. Recall that M

is closed, that Λ is countable, and that N is a countable union of closed sets, so Baire's category Theorem applies, and there exists $\lambda \in \Lambda$ and an open set U of M such that $U \subset N\lambda$, so $UU^{-1} \subset N$. Since M is a connected subgroup, UU^{-1} generates M , so $M \subset N$. \square

The following lemma will be useful both for proving that the closure of $A_1(y_1, y_2)$ is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

Lemma 3.7. *Let F be a closed connected subgroup of $G_1 \times G_2$ such that $F(y_1, y_2)$ contains the closure of $A_1(y_1, y_2)$. Then $F = G_1 \times G_2$.*

Proof. By Lemma 3.2, the set of first coordinates of the set

$$\{(a(s)d_1y_1, a(-s)d_2y_2) : s \geq 0, d_i \in N_i\},$$

is dense in G_1/Γ_1 and the second coordinates lies in the compact set K_2 , so the closure of $A_1(y_1, y_2)$ contains points of arbitrary first coordinate with their second coordinate in K_2 . Consequently, the set of first coordinates of $F(y_1, y_2)$ is the whole G_1/Γ_1 , and similarly for the set of second coordinates. For $i = 1, 2$, Lemma 3.6 now applies to $L = M = G_i$, $\Lambda = \Gamma_i$, $N = p_i(F)$, which is a countable union of closed sets because $G_1 \times G_2$ is σ -compact, and $w = y_i$, and so $p_i(F) = G_i$.

In order to apply Lemma 3.4 and finish the proof, we have to show that $A_1 \subset F$. Again, this follows from a direct application of Lemma 3.6 to $L = G_1 \times G_2$, $\Lambda = \Gamma_1 \times \Gamma_2$, $M = A_1$, $N = F$, $w = (y_1, y_2)$. \square

3.4. Proof of Theorem 1, part (1). We now proceed to proving Theorem 1, part (1). The proof of Proposition 1 is similar and is omitted.

Recall that in this case, we fixed $A = \Psi(A_1)$ and $x = \overline{\Psi}(y_1, y_2)$.

Assume \overline{Ax} is homogeneous, that is $\overline{Ax} = Fx$ for a closed connected subgroup F of G . Since $Ax \subset \overline{\Psi}(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$, which is closed in G/Γ , Lemma 3.6 imply that $F \subset \Psi(G_1 \times G_2)$. By Lemma 3.7, $F = \Psi(G_1 \times G_2)$, so $Fx = G/\Gamma$ and Ax is dense in $\overline{\Psi}(G_1 \times G_2)$, which is a contradiction.

Now assume \overline{Ax} fibers over the orbit of a one-parameter subgroup. Let F be a closed connected subgroup, L a Lie group and $\phi : F \rightarrow L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$, we have $A \subset F'$. By Lemma 3.5, $F'x$ is closed in $Fx \cap \overline{\Psi}(G_1 \times G_2)$, so is closed in G/Γ . By Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily. Let $H = \text{Ker}(\phi \circ \Psi) \subset G_1 \times G_2$, so $A_1/(A_1 \cap H)$ is a one-parameter group by assumption (b).

The subgroup H is a normal subgroup of the semisimple group $G_1 \times G_2$, which has only four kind of normal subgroups : finite, $G_1 \times G_2$, $G_1 \times \text{finite}$ and $\text{finite} \times G_2$. None of these possible normal subgroups have the property that they intersect A_1 in a codimension 1 subgroup, so this is a contradiction.

3.5. The arithmetic lattice. Here we prove that $\mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ is a lattice in $\mathbf{SL}(n, \mathbf{R})$. Let P, Q be the polynomials with coefficients in $\mathbf{Q}(\sqrt{2})$ such that for any $X, Y \in M_n(\mathbf{C})$

$$\det(X + \sqrt[4]{2}Y) = P(X, Y) + \sqrt[4]{2}Q(X, Y).$$

For an integral domain $\mathbf{A} \subset \mathbf{C}$, consider the set of pairs of matrices:

$$\begin{aligned} \mathbf{G}(\mathbf{A}) = \{ (X, Y) \in M_n(\mathbf{A})^2 : {}^tXX - \sqrt{2}{}^tYY = I_n, {}^tXY - {}^tYX = 0, \\ P(X, Y) = 1, Q(X, Y) = 0 \}, \end{aligned}$$

which implies that $({}^tX - \sqrt[4]{2}{}^tY)(X + \sqrt[4]{2}Y) = I_n$ and $\det(X + \sqrt[4]{2}Y) = 1$ for all $(X, Y) \in \mathbf{G}(\mathbf{A})$. Endow $\mathbf{G}(\mathbf{A})$ with the multiplication given by

$$(X, Y)(X', Y') = (XX' + \sqrt{2}YY', XY' + YX'),$$

which is such that the map $\phi : \mathbf{G}(\mathbf{A}) \rightarrow \mathbf{SL}(n, \mathbf{C})$, $(X, Y) \mapsto X + \sqrt[4]{2}Y$ is a morphism. With this structure, \mathbf{G} is an algebraic group, which is clearly defined over $\mathbf{Q}(\sqrt{2})$. Let τ be the nontrivial field automorphism of $\mathbf{Q}(\sqrt{2})/\mathbf{Q}$, it can be checked that the map ϕ is an isomorphism between $\mathbf{G}(\mathbf{R})$ and $\mathbf{SL}(n, \mathbf{R})$, and that moreover $\phi' : \mathbf{G}^\tau(\mathbf{R}) \rightarrow \mathbf{SL}(n, \mathbf{C})$, $(X, Y) \mapsto X + i\sqrt[4]{2}Y$ is an isomorphism onto $\mathbf{SU}(n)$. Let $\mathbf{H} = \text{Res}_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}} \mathbf{G} = \mathbf{G} \times \mathbf{G}^\tau$. Then \mathbf{H} is defined over \mathbf{Q} (see for example [16, 6.1.3], for definition and properties of the restriction of scalars functor). It follows from a Theorem of Borel and Harish-Chandra [16, Theorem 3.1.7] that $\mathbf{H}(\mathbf{Z})$ is a lattice in $\mathbf{H}(\mathbf{R})$. Since $\mathbf{SU}(n)$ is compact, it follows that the projection of $\mathbf{H}(\mathbf{Z})$ onto the first factor of $\mathbf{G}(\mathbf{R}) \times \mathbf{G}^\tau(\mathbf{R})$ is again a lattice. Using the isomorphism between $\mathbf{G}(\mathbf{R})$ and $\mathbf{SL}(n, \mathbf{R})$, this projection can be identified with

$$\mathbf{G}(\mathbf{Z}[\sqrt{2}]) = \mathbf{SU}(n, \mathbf{Z}[\sqrt{2}] + \sqrt[4]{2}\mathbf{Z}[\sqrt{2}], \sigma) = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma).$$

3.6. Proof of Theorem 1, part (2). Note that, as stated implicitly in Section 2.3,

$$\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}) \subset \Gamma \cap M,$$

so $\Gamma \cap M$ is a lattice in M , and $M/(M \cap \Gamma)$ is a closed, A -invariant subset of G/Γ . Notice also that the map Ψ defined by Equation (2) defines an embedding $\overline{\Psi} : G_1 \times G_2/\Gamma_1 \times \Gamma_2 \rightarrow G/\Gamma$.

Assume \overline{Ax} is homogeneous, that is $\overline{Ax} = Fx$ for a closed connected subgroup F of G . Since $Ax \subset M/(M \cap \Gamma)$, which is closed in G/Γ , Lemma 3.6 applied twice gives that $A \subset F \subset M$. Let $F' = F \cap \Psi(G_1 \times G_2)$, again by Lemma 3.5, $F'x$ is a closed subset of $\text{Im}(\overline{\Psi})$. Since $A_1 \subset F'$, $\Psi(A_1)x \subset F'x$ and Lemma 3.7 implies that $F' = \Psi(G_1 \times G_2)$. Since A contains $\varphi(e, e, t)$ for all $t \in \mathbf{R}$, we have $M = AF' \subset F$ so $F = M$ necessarily.

By Lemma 3.3, the A_1 -orbit of (y_1, y_2) is not dense; the topological transitivity of the action of A_1 on $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ implies that moreover the closure of this orbit has empty interior. Thus, the $A_1 \times \mathbf{R}$ -orbit of $(y_1, y_2, 0)$ is also nowhere

dense in $G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}$. The map $\overline{\varphi}$ being a finite covering, the A -orbit of x is nowhere dense. This is a contradiction with $F = M$.

Now assume \overline{Ax} fibers over the orbit of a one-parameter non-**Ad**-unipotent subgroup. Let F be a closed connected subgroup, L a Lie group and $\phi : F \rightarrow L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$ and $F'' = F \cap M$, we have $A_1 \subset F'$ and $A \subset F''$. Similarly, $F'x$ and $F''x$ are closed in G/Γ . Again, by Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily, and like before, $AF' \subset F'' \subset M$ so $F'' = M$.

Let $H = \text{Ker}(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbf{R}$, so $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$ is a one-parameter group. This time, possibilities for the closed normal subgroup H are: finite $\times \Lambda$, $G_1 \times G_2 \times \Lambda$, $G_1 \times \text{finite} \times \Lambda$ and finite $\times G_2 \times \Lambda$, where Λ is a closed subgroup of \mathbf{R} . Of all these possibilities, only $G_1 \times G_2 \times \Lambda$, where Λ is discrete, has the required property that $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$ is a one-parameter group. This proves that $\Psi(G_1 \times G_2) \subset \text{Ker}(\phi)$, so $F \subset N_G(\Psi(G_1 \times G_2))$. However, the normalizer of $\Psi(G_1 \times G_2)$ in G is the group of block matrices having for connected component of the identity the group M . So by connectedness of F , $F \subset M$, and since $M = F'' \subset F$, we have $F = M$. Thus $L = F/\text{Ker}(\phi) = \mathbf{R}/\Lambda$ is abelian, and a fortiori every element of L is unipotent; this contradicts (b).

4. PROOF OF THEOREM 2

The proof of Theorem 2 is divided in two independent lemmas.

Lemma 4.1. *The family $(z_1, \dots, z_{2q}, 1)$ is linearly independent over \mathbf{Q} .*

Proof. Consider a linear combination:

$$\sum_{i=1}^q a_i z_i + b_i z_{i+q} = c.$$

We can assume that a_i, b_i and c are integers. Let $k_0 \geq 1$, write

$$(3) \quad \left(\prod_{i=1}^q p_i \right)^{N^{2k_0+1}} \left(\sum_{i=1}^q \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c \right) = \\ - \left(\prod_{i=1}^q p_i \right)^{N^{2k_0+1}} \left(\sum_{i=1}^q \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} \right).$$

It is clear the left hand side is an integer. Since $1 < p_1 < \dots < p_q$, the right hand side is less in absolute value than

$$\begin{aligned} p_q^{qN^{2k_0+1}} 2q \sup_i (|a_i|, |b_i|) \sum_{k \geq 0} \left(p_1^{-N^{2k_0+2}} \right)^{N^{2k}} \\ \leq 4q \sup_i (|a_i|, |b_i|) p_q^{qN^{2k_0+1}} p_1^{-N^{2k_0+2}} \\ \leq 4q \sup_i (|a_i|, |b_i|) \exp(N^{2k_0+1}(q \log p_q - N \log p_1)). \end{aligned}$$

Since $N > q \frac{\log(p_q)}{\log(p_1)}$, the last expression tends to zero. This proves the right-hand side of (3) is zero for large enough k_0 , so for all large k ,

$$\sum_{i=1}^q a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.$$

The p_i being distincts, this implies that for $i \in \{1, \dots, q\}$, $a_i = b_i = 0$. \square

The following Lemma implies easily that the orbit of z under Ω cannot be dense.

Lemma 4.2. *For all $\epsilon > 0$, there exists $L > 0$, such that for all $n_1, \dots, n_q \geq 0$ with $\sum_{i=1}^q n_i \geq L$, there exists $j \in \{1, \dots, 2q\}$ such that $p_1^{n_1} \dots p_q^{n_q} z_j$ lies in the interval $[0, \epsilon]$ modulo 1.*

Proof. Let $s \in \{1, \dots, q\}$ such that for all $r \in \{1, \dots, q\}$, $p_s^{n_s} \geq p_r^{n_r}$. Let k_0 be the integer part of $\log(n_s)/2 \log(N)$, then either $N^{2k_0} \leq n_s \leq N^{2k_0+1}$, or $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$. In the first case, take $j = s$, then:

$$p_1^{n_1} \dots p_q^{n_q} z_j = p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq 1} p_s^{-N^{2k}} = p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \pmod{1}.$$

We have

$$\sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{-N^{2k_0+2}},$$

so, using the fact that for all $r \in \{1, \dots, q\}$, $p_r^{n_r} \leq p_s^{n_s} \leq p_s^{N^{2k_0+1}}$, we obtain:

$$p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{qN^{2k_0+1} - N^{2k_0+2}} \leq 2p_s^{N^{2k_0+1}(q-N)},$$

but by hypothesis we have $N > q \frac{\log(p_q)}{\log(p_1)} > q$, so the preceding bound is small whenever k_0 is large. Because of the definition of k_0 , we have

$$k_0 \geq \frac{\log \frac{\sum_{i=1}^q n_i \log p_i}{q \log p_q}}{2 \log N} \geq \frac{\log \frac{L \log p_1}{q \log p_q}}{2 \log N},$$

so k_0 is arbitrary large when L is large.

In the second case $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$, one can proceed similarly with $j = s + q$.

□

5. ACKNOWLEDGEMENTS

I am indebted to Yves Guivarc'h for mentioning to me the problem of orbit closure in the toral endomorphisms setting, and to Livio Flaminio for numerous useful comments. I also thank Françoise Dal'Bo, and Sébastien Gouëzel for stimulating conversations on this topic.

REFERENCES

- [1] D. Berend, *Minimal sets on tori*, Ergodic Theory Dyn. Syst. 4, 499-507 (1984).
- [2] A. Borel, *Introduction aux groupes arithmétiques*, Publications de l'Institut de Mathématiques de l'Université de Strasbourg XV, Hermann, 1969.
- [3] M. Einsiedler, A. Katok and E. Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*. Ann. of Math. (2) 164 (2006), no. 2, 513-560.
- [4] D. Ferte, *Dynamique topologique d'une action de groupe sur un espace homogène : exemples d'actions unipotente et diagonale*, PhD Thesis, Rennes I.
- [5] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*. Math. Syst. Theory 1, 1-49 (1967).
- [6] D. Kleinbock, N. Shah, and A. Starkov, *Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory*. Handbook of dynamical systems, Vol. 1A, 813-930, North-Holland, Amsterdam, 2002.
- [7] E. Lindenstrauss and B. Weiss, *On sets invariant under the action of the diagonal group*, Ergodic Theory Dyn. Syst. 21, No.5, 1481-1500 (2001).
- [8] G. Margulis, *Problems and conjectures in rigidity theory*, Mathematics : Frontiers and Perspective, 161-174, Amer. Math. Soc., Providence, RI, 2000.
- [9] G. Margulis and N. Qian, *Rigidity of weakly hyperbolic actions of higher real rank semisimple Lie groups and their lattices*, Ergodic Theory Dyn. Syst. 21, No.1, 121-164 (2001).
- [10] D. Meiri and Y. Peres, *Bi-invariant sets and measures have integer Hausdorff dimension*, Ergodic Theory Dyn. Syst. 19 (1999), 523-534.
- [11] G. Prasad and M.S. Raghunathan, *Cartan subgroups and lattices in semi-simple groups*, Ann. Math. (2) 96, 296-317 (1972).
- [12] G. Prasad and A.S. Rapinchuk, *Irreducible tori in semisimple groups*, International Mathematics Research Notices 2001(23), 1229-1242. Erratum, 2002 (17), 919-921
- [13] M. Ratner, *Raghunathan's topological conjecture and distribution of unipotent flows*, Duke Mathematical Journal 63 (1), 1991, 235-280.
- [14] N. Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Mathematische Annalen 289, 325-334 (1991).
- [15] G. Tomanov, *Actions of maximal tori on homogeneous spaces*. Rigidity in dynamics and geometry (Cambridge, 2000), 407-424, Springer, Berlin, 2002.
- [16] R.J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics vol 81, Birkhäuser, 1984.

UNIVERSITÉ RENNES I, IRMAR, CAMPUS DE BEAULIEU 35042 RENNES CEDEX - FRANCE
E-mail address: francois.maucourant@univ-rennes1.fr